Minimum Degree Ordering

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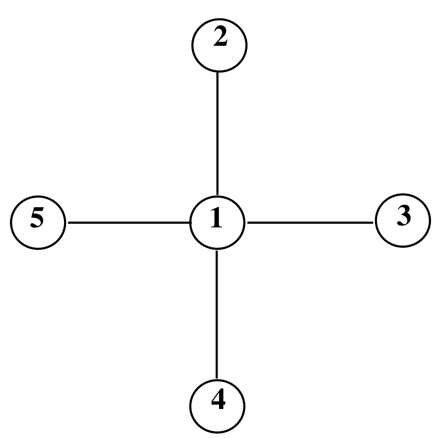
Outline

- Review of elimination graphs
  - Elimination graphs
  - Quotient graphs
- The Markowitz procedure
- The Minimum Degree ordering heuristic
- Improving the basic minimum degree algorithm
- Tie-breaking
- Future directions

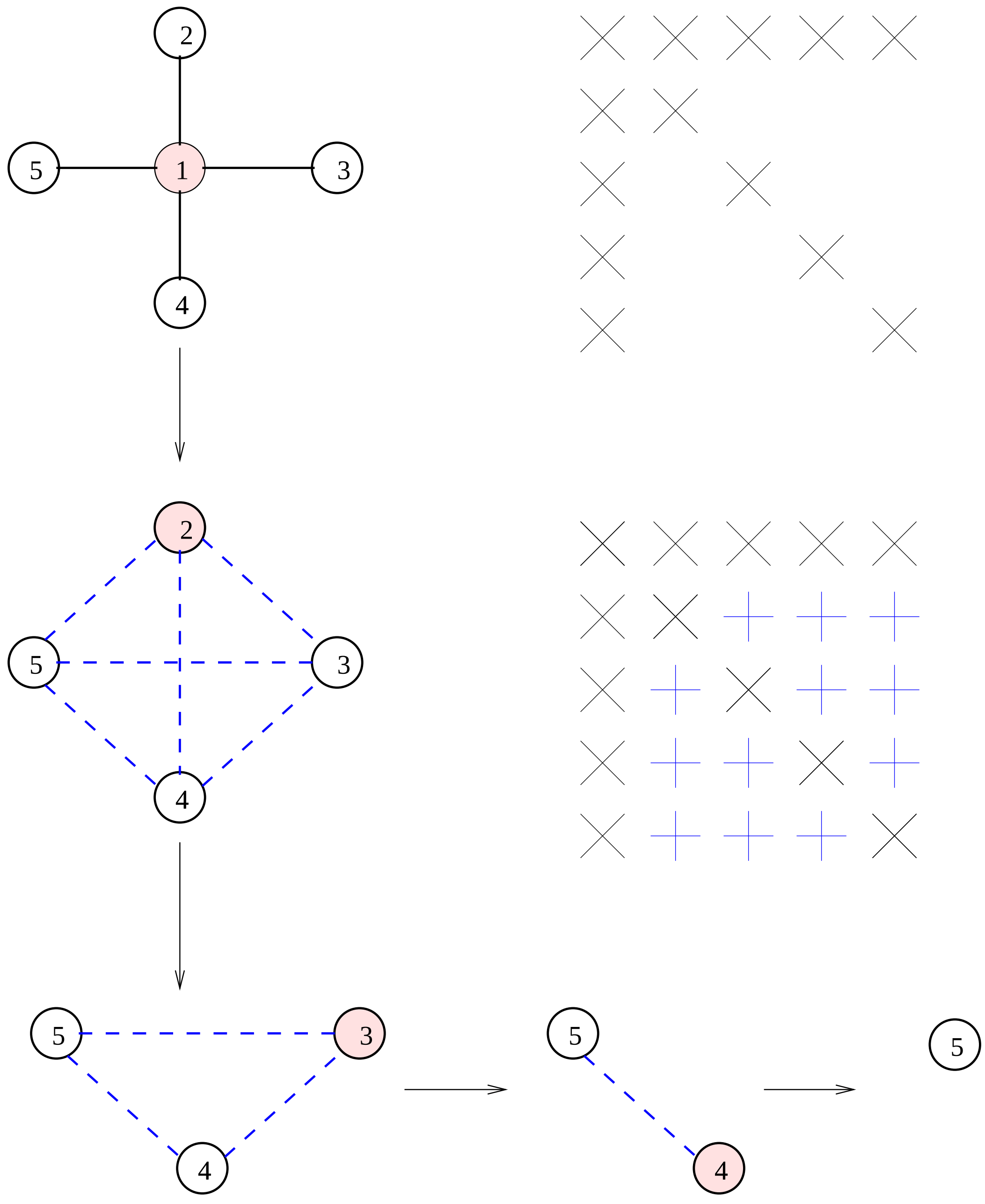
Elimination Graph Model

- Let $G^0 = G = (X, E)$. The sequence of elimination graphs is given by $G_0 \longrightarrow G_1 \longrightarrow \cdots G_{n-1}$. The i-th elimination graph is formed by:
  - deleting node x_i and its incident edges
  - adding edges to the graph so that nodes in $Adj(x_i)$ are pairwise adjacent (i.e., form a clique)
- The filled graph is designated $G^F = (X^F, E^F)$, where $F = L + L^T$. E^F consists of all edges in E^A together with all the edges added during the factorization.
- *The unordered pair* $\{x_i, x_j\} \in E^F$ *if and only if* $\{x_i, x_j\} \in E^A$ *or* $\{x_k, x_j\} \in E^F$ *for some* $k < \min\{i, j\}$.
- Arrow matrix example:

$$\begin{bmatrix} X & X & X & X & X \\ X & X & & & \\ X & & X & & \\ X & & & X & \\ X & & & & X \end{bmatrix}$$

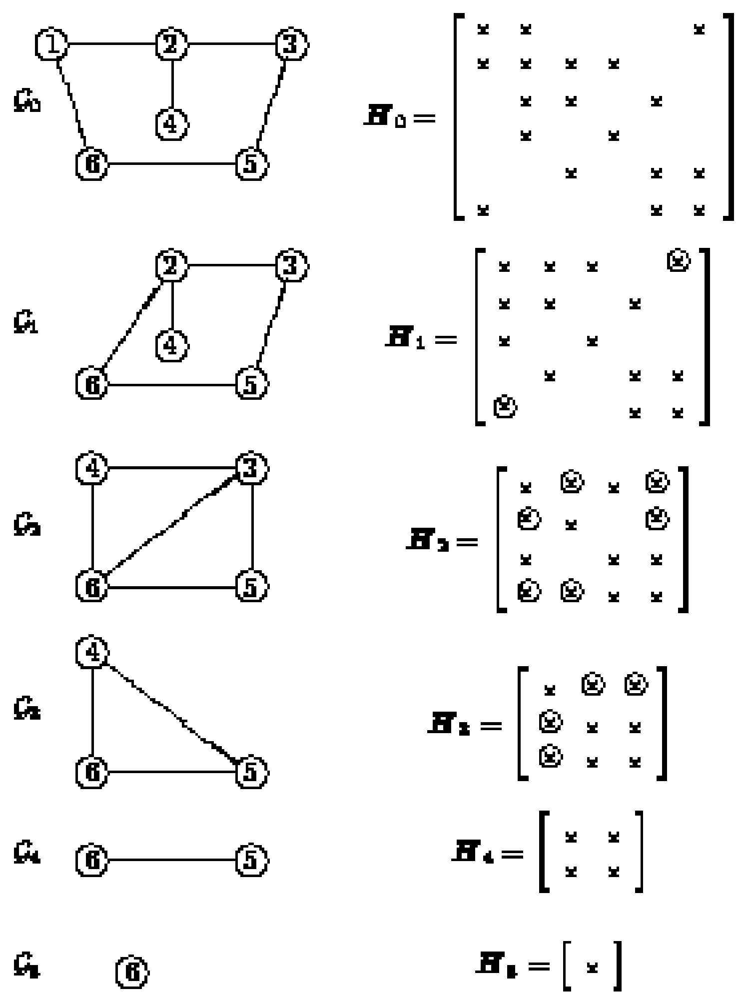


Elimination Graph Example



Another Elimination Graph Example

George and Liu, page 111.



Reachable Sets

- The recursively defined sequence of elimination graphs can be modeled by using the notion of *reachable sets*.

- The node x is said to be *reachable* from a node y *through* S if there exists a path $(y, v_1, \cdots, v_k, x)$ from y to x such that $v_i \in S$ for $1 \leq i \leq k$.

- The reachable set of y through S, denoted $Reach(y, S)$, is defined to be:

$$Reach(y, S) = \{x \notin S \mid x \text{ is reachable from } y \text{ through } S.$$

- For example, if $S = \{x_2, x_1\}$, then

$$Reach(x_4, S) = x_6$$

- The filled graph G^F is fully characterized by reachable sets by the following theorem:

$$E^F = \{\{x_i, x_j\} \mid x_j \in Reach(x_i, \{x_1, x_2, \cdots, x_{i-1}\})\}.$$

- Given the above definition and a node y in the elimination graph $G_i = (X_i, E_i)$, the set of nodes adjacent to y in G_i is given by

$$Reach(y, \{x_1, \cdots, x_i\}).$$

The Markowitz Procedure

- Amount of fill is sensitive to the order in which rows and columns of the matrix are processed.

- Reordering goal: limit the fill during factorization.

- Markowitz (Management Science, 1957) had the following idea:
  - Suppose the elimination has proceeded through the first k stages of Gaussian elimination.
  - For each row i in the active $(n-k) \times (n-k)$ submatrix:
    * Let $r_i^{(k)}$ denote the number of entries in row i.
    * Let $c_j^{(k)}$ denote the number of entries in column j.
    * Select the entry $a_{ij}^{(k)}$ from the $(n-k) \times (n-k)$ that is not too small numerically and that minimizes the expression
    $$(r_i^{(k)} - 1)(c_j^{(k)} - 1).$$
  - At each elimination stage, we need to know the updated sparsity pattern of the reduced $(n-k) \times (n-k)$ submatrix.
  - We must be able to access rows **and** columns.
  - The numerical values are required to judge acceptability of pivot.

Basic Minimum Degree Algorithm

Consider the case when the matrix is symmetric positive definite, thus there is no need to pivot for stability.

begin
  $G \longleftarrow$ given symmetric graph
  while $G \neq \emptyset$ **do**
    select a node y of minimum degree in G and order y next
    $G \longleftarrow G_y$
  end
end

- Given a node $v \in G$, we observe the following relation between $degree_{G_y}(v)$ and $degree_G(v)$:
  - Observation 1. *If v is not adjacent to y, then $degree_{G_y}(v) = degree_G(v)$.*
  - Observation 2. *If v is adjacent to y, then*

$$degree_{G_y}(v) \geq degree_G(v) - 1 \geq degree_G(y) - 1$$

- The quality of the minimum degree ordering depends on the tie-breaking strategy.
- Finding the optimal ordering is an NP-complete problem (Yannakakis, 1981).

Improvements

- Most important step of MD algorithm: $G \longleftarrow G_y$.
  - accommodating fill caused by removal of y
  - recalculation of degrees of nodes adjacent to y; this is the most time consuming step in the entire algorithm.
- Possible improvements:
  - Mass elimination of indistinguishable nodes
  - Representation of elimination graphs
  - Incomplete degree update
  - Element absorption
  - Multiple elimination
  - External degree

Mass Elimination

- If node y is selected as the minimum degree node in the elimination graph G, then the subset:

$$Y = \{z \in Adj_G(y) \mid degree_{G_y}(z) = degree_G(y) - 1\}$$

can be selected next (in any order) in the minimum degree algorithm.

- During one step of the graph elimination algorithm, instead of a single node, a node set of cardinality $|Y \cup \{y\}|$ is eliminated. This set of nodes is sometimes referred to as *supernode, supervariable*, or *prototype node.*
- The elements of Y must be *indistinguishable* with respect to elimination.

Indistinguishable Nodes:

- Two nodes u and v are indistinguishable with respect to elimination in G if

$$Adj_G(u) \cup \{u\} = Adj_G(v) \cup \{v\}$$

or

$Reach(u, S) \cup \{u\} = Reach(v, S) \cup \{v\}$ where S is the subset of eliminated nodes.

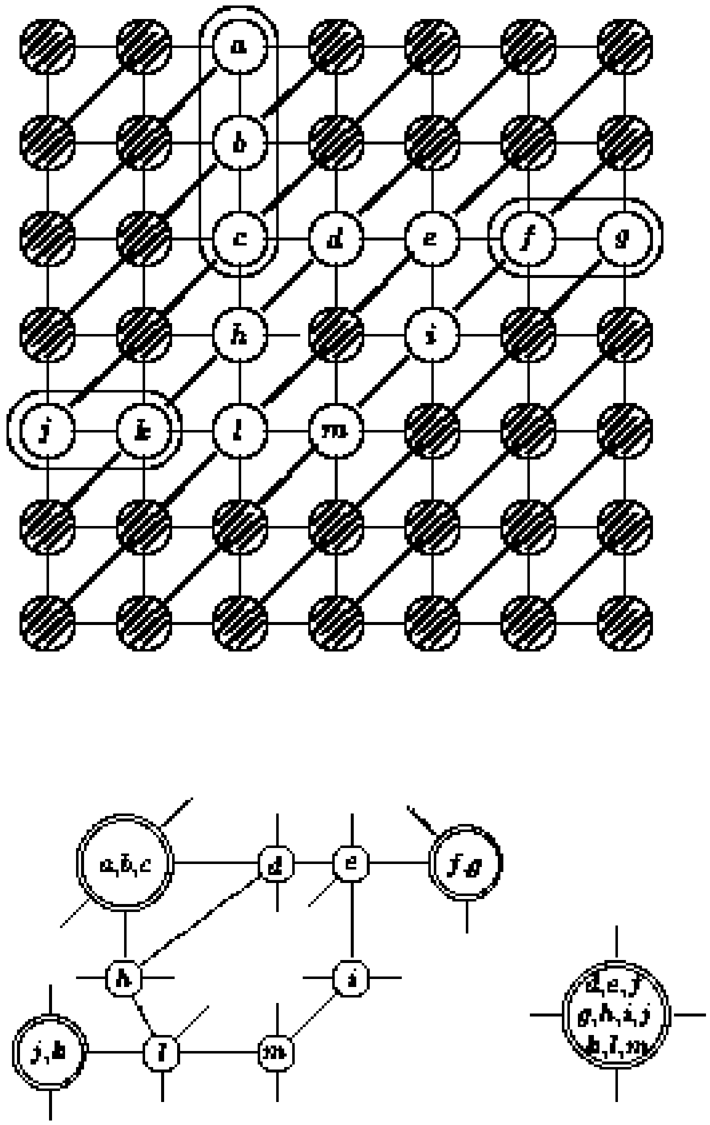
- The set $Adj_G(u) \cup \{u\}$ is referred to as the *neighborhood set* of u.

- If u and v are indistinguishable with respect to the subset S, then for $T \supset S$

$$|Reach(u, T)| = |Reach(v, T)|.$$

- In other words, nodes u and v can be eliminated together since after the elimination of node u, node v becomes a node of minimum degree.

Indistinguishable Nodes Example



Representation of Elimination Graphs

- The elimination graph transformation "$G \longleftarrow G_y$" involves both deletion and addition of edges.

- Use a representational scheme based on a view of graphs as a set of cliques, or generalized elements. Initially, there are $|E|$ cliques.
- The elimination transformation is performed as follows.
  Let $\{K_1, K_2, \cdots, K_q\}$ be the set of cliques for the current graph G. Assume y is a node of minimum degree. Let $\{K_{S_1}, \cdots, K_{S_t}\}$ be the subset of cliques to which y belongs. The elimination graph requires two steps:
  1. Remove the cliques $K_{S_1}, \cdots, K_{S_t}$ from$\{K_1, K_2, \cdots, K_q\}$.
  2. Add the new clique $K = (K_{S_1} \cup \cdots \cup K_{S_t}) - \{y\}$ into the clique set.
- Theorem: $|K| < \sum_{i=1}^{t} |K_{S_i}|$.

- In other words, the amount of storage required will *never* exceed the amount needed to represent the original graph since the set of cliques never grows beyond its original size.

Incomplete Degree Update

- Given two nodes u and v in G, the node v is said to be *outmatched* by u if

$$Adj_G(u) \cup \{u\} \subseteq Adj_G(v) \cup \{v\}.$$

- If u outmatches v, $degree_G(u) \leq degree_G(v)$. This property is preserved under elimination graph transformation.

- Theorem: If a node v is outmatched by u in G, it is also outmatched by u in the graph G_y.

- If a node v becomes outmatched by u in the elimination process, the node u can be eliminated before v in the minimum degree algorithm, in other words, it is not necessary to update the degree of v until the node u has been eliminated.

- Using the outmatched property, degree updates are skipped for those nodes, which will not participate in the next round of minimum degree selection.

Element Absorption

- Store cliques instead of edges,
- Let $\{K_1, K_2, \cdots, K_q\}$ be the set of cliques for the current graph G.
  - If $K_s \subseteq K_t$ for some s and t, then the graph G can be represented by the clique set $\{K_1, K_2, \cdots, K_q\} - K_s$.
- Clique K_s can be viewed as being *absorbed* by clique K_t.
- This approach removes any detected cliques redundancy in the representation.
- Reducing the number of cliques will speed up the ordering process since there is less overhead in manipulating clique sets.

Multiple Elimination

- Instead of performing it after each minimum degree node selection and graph transformation, postpone the degree update to a later stage.

- Before a degree update is performed:
  - Select a node y of minimum degree; its elimination will not affect the subgraph $G - (adj_G(y) \cup \{y\})$.
  - Repeat the previous step until no nodes of degree $degree_G(y)$ remain.

- Perform a degree update.

- In essence, before each degree update step, an *independent* set of nodes with minimum degree is selected. The fact that they are independent allows the delay of degree update until the entire set is determined and eliminated.

External Degree

- Instead of using the true degree of a node, use the *external* degree, which is the number of nodes adjacent to it that are **not** indistinguishable from itself.

- When the technique of mass elimination is used, the size of the resulting clique is the same as the external degree of the vertices eliminated.

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ 20 & 21 & 22 & 16 & 15 \\ \bullet & \bullet & 23 & \bullet & \bullet \\ \bullet & 28 & 24 & 25 & \bullet \\ \bullet & 17 & \bullet & 19 & \bullet \end{bmatrix}$$

Tie-Breaking Strategies

- Let A be the matrix associated with a $k \times k$ regular grid.
- The best ordering on A would yield a Cholesky factor with at least $O(k^2 log_2 k)$ nonzeros.

- One such perfect ordering is nested dissection on a grid.
- It is possible for the minimum degree algorithm to produce an ordering for which the resulting Cholesky factor has $O(k^{2 log_3 4})$ nonzeros.

- Two popular tie-breaking strategies:
  - Random tie-breaking (Harwell MA27 code, Waterloo SPARSPAK package, Yale sparse matrix package (YSMP).
  - The multiple elimination technique provides a kind of tie-breaking strategy.
  - Preordering: rearrange the adjacency structure before passing it to the ordering routine, e.g. use reverse Cuthill-McKee preordering.

Summary

- The symmetric minimum degree algorithm implements a heuristic for minimizing the fill.
- Several enhancements to the basic algorithm aim to improve storage and increase performance of the ordering phase.

References

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Chapter 7 of I. Duff, A. Erisman and J. Reid, *Direct Methods for Sparse Matrices*, Oxford University Press, New York, 1986.

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